

# Using Quantum Oblivious Transfer to Cheat Sensitive Quantum Bit Commitment

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## Abstract

It is well known that unconditionally secure bit commitment is impossible even in the quantum world. In this paper a weak variant of quantum bit commitment, introduced independently by Aharonov et al. [2] and Hardy and Kent [8] is investigated. In this variant, the parties require some nonzero probability of detecting a cheating, i.e. if Bob, who commits a bit  $b$  to Alice, changes his mind during the revealing phase then Alice detects the cheating with a positive probability (we call this property *binding*); and if Alice gains information about the committed bit before the revealing phase then Bob discovers this with positive probability (*sealing*). In our paper we give quantum bit commitment scheme that is simultaneously binding and sealing and we show that if a cheating gives  $\varepsilon$  advantage to a malicious Alice then Bob can detect the cheating with a probability  $\Omega(\varepsilon^2)$ . If Bob cheats then Alice's probability of detecting the cheating is greater than some fixed constant  $\lambda > 0$ . This improves the probabilities of cheating detections shown by Hardy and Kent and the scheme by Aharonov et al. who presented a protocol that is either binding or sealing, but not simultaneously both.

To construct a cheat sensitive quantum bit commitment scheme we use a protocol for a weak quantum one-out-of-two oblivious transfer ( $\binom{2}{1}$ -OT). In this version, similarly as in the standard definition, Alice has initially secret bits  $a_0, a_1$  and Bob has a secret selection bit  $i$  and if both parties are honest they solve the  $\binom{2}{1}$ -OT problem fulfilling the standard security requirements. However, if Alice is dishonest and she gains some information about the secret selection bit then the probability that Bob computes the correct value is proportionally small. Moreover, if Bob is dishonest and he learns something about both bits, then he is not able to gain full information about one of them.

## 1 Introduction

In bit commitment protocol Bob commits a bit  $b$  to Alice in such a way that Alice learns nothing (in an information theoretic sense) about  $b$  during this phase and later on, in the revealing time, Bob cannot change his mind. It is well known that unconditionally secure bit commitment is impossible even when the parties use quantum communication protocols ([10, 11]). Thus, much effort has been focused on schemes using some weakened security assumptions.

In a weak variant of quantum bit commitment, introduced independently by Aharonov et al. [2] and Hardy and Kent [8], the protocol should guarantee that if one party cheats then the other has good probability of detecting the mistrustful party. Speaking more precisely, we require that if Bob changes his mind during the revealing phase then Alice detects the cheating with

a positive probability (we call this property *binding*) and if Alice learns information about the committed bit before the revealing time then Bob discovers the leakage of information with positive probability (*sealing* property).

In [8] Hardy and Kent give protocol that is simultaneously sealing and binding and prove that if Alice (Bob) uses a strategy giving  $\varepsilon > 0$  advantage then Bob (Alice, resp.) can detect the cheating with a probability strictly greater than 0. The authors do not analyze, however, the quantitative dependence of the probability on  $\varepsilon$ . In [2] Aharonov et al. present a similar protocol to that proposed in [8] such that after depositing phase either Alice or Bob challenges the other party and (1) when Alice asks Bob to reveal  $b$  and Bob influences the value with advantage  $\varepsilon$  then she detects the cheating with probability  $\Omega(\varepsilon^2)$  and (2) when Bob challenges Alice to return the depositing qubit and Alice predicts  $b$  with advantage  $\varepsilon$  then Bob detects the cheating with probability  $\Omega(\varepsilon^2)$ . Thus the protocol is either binding or sealing, but not simultaneously both (the authors therefore call the protocol a quantum bit escrow). Aharonov et al. left open whether simultaneous binding and sealing can be achieved.

In our paper we give the first, up to our knowledge, QBC scheme that is simultaneously binding and sealing such that if Alice's cheating gives  $\varepsilon$  advantage then Bob can detect the cheating with a probability which is  $\Omega(\varepsilon^2)$ . If Bob cheats (anyhow) then Alice's probability of detecting the cheating is greater than some fixed constant  $\lambda > 0$ , i.e. when Bob decides to set the value  $b$  to 0 or to 1 and in the revealing time wants to change his mind then for any strategy Bob uses the probability that Alice detects this attack is greater than  $\lambda$ . To construct such scheme we use a protocol for a weak variant of quantum oblivious transfer.

## 1.1 Our Contribution

In the one-out-of-two oblivious transfer problem ( $\binom{2}{1}$ -OT, for short) Alice has initially two secret bits  $a_0, a_1$  and Bob has a secret selection bit  $i$ . The aim of a  $\binom{2}{1}$ -OT protocol is disclosing the selected bit  $a_i$  to Bob, in such a way that Bob gains no further information about the other bit and Alice learns nothing at all. The problem has been proposed by Even et al. [7], as a generalization of Rabin's notion for oblivious transfer [12]. Oblivious transfer is a primitive of central importance particularly in secure two-party and multi-party computations. It is well known ([9, 4]) that  $\binom{2}{1}$ -OT can be used as a basic component to construct protocols solving more sophisticated tasks of secure computations such as two-party oblivious circuit evaluation. Several secure OT protocols have been proposed in the literature [3, 5, 6] however, even in quantum world, there exists no unconditionally secure protocol for  $\binom{2}{1}$ -OT (see e.g. [11]).

In this paper we define a weak variant of one-out-of-two oblivious transfer. Similarly as in the standard definition, in a weak  $\binom{2}{1}$ -OT protocol Alice has initially secret bits  $a_0, a_1$  and Bob has a secret selection bit  $i$  and if both parties are honest<sup>1</sup> they solve the  $\binom{2}{1}$ -OT problem fulfilling the standard requirements. However if Alice is dishonest and she gains some information about the secret selection bit then the probability that Bob computes the correct value is proportionally decreased. Moreover, if Bob is dishonest he can learn about both bits, but if he does so then he is not able to gain full information about one of them.

In the paper we present a weak  $\binom{2}{1}$ -OT protocol which, speaking informally (precise definitions will be given in Section 3), fulfills the following properties.

- If both Alice having initially bits  $a_0, a_1$  and Bob having bit  $i$  are honest then Bob learns the selected bit  $a_i$ , but he gains no further information about the other bit and Alice learns nothing.

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<sup>1</sup>We say that a party is honest if it never deviates from the given protocol.

- If Bob is honest and has a bit  $i$  and Alice learns  $i$  with advantage  $\varepsilon$  then for all  $a_0, a_1 \in \{0, 1\}$  the probability that Bob computes the correct value  $a_i$ , when the protocol completes, is at most  $1 - \Omega(\varepsilon^2)$ .
- If Alice is honest and has bits  $a_0, a_1$  then for every  $i \in \{0, 1\}$  it is true that if Bob can predict the value  $a_{1-i}$  with advantage  $\varepsilon$  then the probability that Bob learns correctly  $a_i$  is at most  $1 - \Omega(\varepsilon^2)$ .

The protocol can be used e.g. by the mistrustful parties for which computing the correct result of  $\binom{2}{1}$ -OT is much more preferential than gaining addition information. In this paper we show an application of the protocol for parties who require some nonzero probability of detecting a cheating. Let us consider the following bit commitment protocol, where  $v := OT((a_0, a_1), i)$  means, for short, that Alice having initially  $a_0, a_1$  and Bob knowing  $i$  perform the weak  $\binom{2}{1}$ -OT protocol and when the protocol completes Bob knows the result  $v$ .

**Protocol 1 (Cheat sensitive QBC)** *B commits bit b;*

- **Depositing phase**
  1. *A chooses randomly bits  $a_0, a_1, a_2, a_3$ ; B chooses randomly bits  $b'$  and  $c$ ;*
  2. *A and B compute*

$$v_0 := OT((a_0, a_1), b'); v_1 := OT((a_2, a_3), b) \text{ if } c = 0 \text{ or}$$

$$v_0 := OT((a_0, a_1), b); v_1 := OT((a_2, a_3), b') \text{ if } c = 1.$$
  3. *B reveals c.*
- **Revealing phase** *B reveals b;*
  - *Sealing test: A sends to B  $a_{2c}, a_{2c+1}$ ; B rejects when  $v_c \neq OT((a_{2c}, a_{2c+1}), b')$ .*
  - *Binding test: B sends to A  $v_{1-c}$ ; A rejects when  $v_{1-c} \neq OT((a_{2-2c}, a_{3-2c}), b)$ .*

One of the main results of this paper says that using our weak  $\binom{2}{1}$ -OT protocol, the bit commitment protocol above has the following properties: (1) If both Alice and Bob are honest, then before revealing time Alice gains no information about  $b$  and at the revealing phase both Bob and Alice accept; (2) if Alice learns  $b$  with advantage  $\varepsilon$  then Bob detects cheating with probability  $\Omega(\varepsilon^2)$ , and (3) if Bob tries to change  $b$  during the revealing phase then for any strategy he uses the probability that Alice detects the cheating is greater than some positive constant.

The paper is organized as follows. In Section 2 some basic quantum preliminaries are given. In Section 3 we define formally properties of a weak  $\binom{2}{1}$ -OT protocol and prove that the given scheme fulfills the properties. Section 4 gives formal definition of binding and sealing and proves that Protocol 1 is simultaneously binding and sealing.

## 2 Preliminaries

The model of two-party computation we use in this paper is essentially the same as defined in [2]. We assume that the reader is already familiar with basics of quantum cryptography (see [2] for an exemplary summary of results that will be used in the following).

Let  $|0\rangle, |1\rangle$  be an encoding of classical bits in our computational (perpendicular) basis. Let  $|0_\times\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ ,  $|1_\times\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$  be an encoding of classical bits in diagonal basis. By  $R_\alpha$ ,  $\alpha \in \{0, \frac{1}{2}, 1\}$ , we denote the unitary operation of rotation by an angle of  $\alpha \cdot \pi/2$ . More formally:

$$R_\alpha := \begin{pmatrix} \cos(\alpha \cdot \frac{\pi}{2}) & \sin(\alpha \cdot \frac{\pi}{2}) \\ -\sin(\alpha \cdot \frac{\pi}{2}) & \cos(\alpha \cdot \frac{\pi}{2}) \end{pmatrix}$$

We should note that this operation allows us to exchange between the bit encoding in perpendicular and in diagonal basis. Moreover, by applying  $R_1$  we can flip the value of the bit encoded in any of those two bases.

For a mixed quantum state  $\rho$  and a measurement  $\mathcal{O}$  on  $\rho$ , let  $\rho^{\mathcal{O}}$  denote the classical distribution on the possible results obtained by measuring  $\rho$  according to  $\mathcal{O}$ , i.e.  $\rho^{\mathcal{O}}$  is some distribution  $p_1, \dots, p_t$  where  $p_i$  denotes the probability that we get result  $i$ . We use  $L_1$ -norm to measure distance between two probability distributions  $p = (p_1, \dots, p_t)$  and  $q = (q_1, \dots, q_t)$  over  $\{1, 2, \dots, t\}$ :  $|p - q|_1 = \frac{1}{2} \sum_{i=1}^t |p_i - q_i|$ .

Let  $\|A\|_t = \text{tr}(\sqrt{A^\dagger A})$ , where  $\text{tr}(A)$  denotes trace of matrix  $A$ . A fundamental theorem gives us a bound on  $L_1$ -norm for the probability distributions on the measurement results:

**Theorem 1 (see [1])** *Let  $\rho_0, \rho_1$  be two density matrices on the same Hilbert space  $\mathcal{H}$ . Then for any generalized measurement  $\mathcal{O}$   $|\rho_0^{\mathcal{O}} - \rho_1^{\mathcal{O}}|_1 \leq \frac{1}{2} \|\rho_0 - \rho_1\|_t$ . This bound is tight and the orthogonal measurement  $\mathcal{O}$  that projects a state on the eigenvectors of  $\rho_0 - \rho_1$  achieves it.*

A well-known result states that if  $|\phi_1\rangle, |\phi_2\rangle$  are pure states, then  $\| |\phi_1\rangle\langle\phi_1| - |\phi_2\rangle\langle\phi_2| \|_t = 2\sqrt{1 - |\langle\phi_1|\phi_2\rangle|^2}$ .

**Lemma 1** *Suppose Bob has a bit  $b$  s.t.  $\Pr[b = 0] = 1/2$  and let Alice generate a state with two quantum registers. Assume she sends the second register to Bob, then Bob depending on  $b$  makes some transformation on his part and sends the result back to Alice. Denote by  $\rho_0$  density matrix of the resulting state for  $b = 0$  and by  $\rho_1$  density matrix of the state for  $b = 1$ . Then for any measurement  $\mathcal{O}$  Alice makes and a value  $v$  Alice learns we have  $\Pr_{b \in R\{0,1\}}[v = b] \leq 1/2 + \frac{|\rho_0^{\mathcal{O}} - \rho_1^{\mathcal{O}}|_1}{2}$ .*

The proof of this lemma follows by some straight forward calculations and will be skipped in this extended abstract. We will use some obvious variations of this lemma to bound the advantage of Alice resp. Bob in what will follow.

### 3 Weak Oblivious Transfer

In this section we give the formal definition of the weak  $\binom{2}{1}$ -OT protocol and then present protocol for this problem.

**Definition 1** *We say that a two-party quantum protocol between Alice and Bob is a  $(\delta, \varepsilon)$ -weak  $\binom{2}{1}$ -OT protocol if the following requirements hold.*

- If both Alice depositing initially bits  $a_0, a_1$  and Bob having bit  $i$  are honest then Bob learns the selected bit  $a_i$  but in such a way that he gains no further information about the other bit and Alice learns nothing.
- Whenever Bob is honest and has a selection bit  $i$ , with  $\Pr[i = 0] = 1/2$ , then for every strategy used by Alice, every value  $i'$  Alice learns about  $i$  and for any value  $a'$  Bob learns at the end of the computation it holds that for all  $a_0, a_1 \in \{0, 1\}$

$$\text{if } \Pr_{i \in R\{0,1\}}[i' = i] \geq 1/2 + \delta \text{ then } \Pr_{i \in R\{0,1\}}[a' = a_i] \leq 1 - \varepsilon.$$

- Whenever Alice is honest and deposits bits  $a_0, a_1$ , with  $\Pr[a_i = 0] = 1/2$ , then for every strategy used by Bob, all values  $a'_0, a'_1$  Bob learns about  $a_0, a_1$ , resp. it holds that for all  $i \in \{0, 1\}$  if  $\Pr_{a_0, a_1 \in R\{0,1\}}[a'_{1-i} = a_{1-i}] \geq 1/2 + \delta$  then  $\Pr_{a_0, a_1 \in R\{0,1\}}[a'_i = a_i] \leq 1 - \varepsilon$ .

**Protocol 2** ( $\binom{2}{1}$ -OT function) Input  $A : a_0, a_1 \in \{0, 1\}$ ,  $B : i \in \{0, 1\}$ ; Output  $B : a_i$ .

1.  $A$  chooses randomly  $\alpha \in_R \{0, \frac{1}{2}\}$  and  $h \in_R \{0, 1\}$  and sends to  $B$ :

$$R_\alpha|a_1 \oplus h\rangle \otimes R_\alpha|a_0 \oplus h\rangle$$

2.  $B$  receives  $|\Phi_1\rangle \otimes |\Phi_0\rangle$ , chooses randomly  $\beta \in_R \{0, 1\}$  and sends  $R_\beta|\Phi_i\rangle$  back to  $A$ .
3.  $A$  receives  $|\Phi\rangle$ , computes  $R_\alpha^{-1}|\Phi\rangle$ , measures the state in computational basis obtaining the result  $n$  and sends  $m = n \oplus h$  to  $B$ .
4.  $B$  receives  $m$  and computes  $a_i = m \oplus \beta$ .

Here, as usually,  $\otimes$  denotes xor. Note that this protocol computes  $\binom{2}{1}$ -OT correctly if both parties are honest. We will now focus on the question whether Protocol 2 still retains security if we use it against malicious parties. The following theorem follows from Lemma 2 and 3 which will be proven in the remaining part of this section:

**Theorem 2** *Protocol 2 is  $(O(\sqrt[2]{\varepsilon}), \varepsilon)$ -weak  $\binom{2}{1}$ -OT protocol.*

### 3.1 Malicious Alice

**Lemma 2** *Let Alice and Bob perform Protocol 2 and assume Bob is honest and deposits a bit  $i$ , with  $\Pr[i = 0] = 1/2$ . Then for every strategy used by Alice, every value  $i'$  Alice learns about  $i$  and for any value  $a'$  Bob learns at the end of the computation it holds that for all  $a_0, a_1 \in \{0, 1\}$  if  $\Pr_{i \in \{0, 1\}}[a' = a_i] \geq 1 - \varepsilon$  then  $\Pr_{i \in \{0, 1\}}[i' = i] \leq 1/2 + 16\sqrt{\varepsilon}$ .*

*Proof:* Any cheating strategy  $\mathcal{A}$  of Alice can be described as preparing some state  $|\Phi\rangle = \sum_{x \in \{0, 1\}^2} |v_x, x\rangle$ , sending the two rightmost qubits to Bob and perform some measurement  $\{H_0, H_1, H_2, H_3\}$  on this what she gets back after Bob's round, where  $H_0, H_1, H_2, H_3$  are four pairwise orthogonal subspaces being a division of whole Hilbert space that comes into play, such that, for  $l, k = 0, 1$ , if our measurement indicates the outcome corresponding to  $H_{2k+l}$  then it reflects Alice's belief that  $i = l$  and that the message  $m = k$  should be sent to Bob.

Assume now, that  $a_0 \oplus a_1 = 0$ . We should note that in this case  $m \oplus a_0 = \beta$ . So Alice, in order to ensure the correct result of the protocol, has to indicate the value of  $\beta$ . Let  $|S\rangle = |v_{00}, 00\rangle + |v_{11}, 11\rangle$ ,  $|A\rangle = |v_{01}, 01\rangle + |v_{10}, 10\rangle$ . That is,  $|S\rangle$  is a part of the state that is symmetric with respect to qubits being sent to Bob and  $|A\rangle$  is the rest being anti-symmetric.

Let  $\rho_{a,b}$  be a density matrix of Alice's system after Bob's round, corresponding to  $i = a$  and  $\beta = b$ . After some calculations we get:

$$\begin{aligned} \rho_{0,0} &= \sum_{x=(x_1,x_2) \in \{0,1\}^2} |v_x x_1\rangle \langle v_x x_1| \\ &\quad + |v_{00}0\rangle \langle v_{10}1| + |v_{10}1\rangle \langle v_{00}0| + |v_{11}1\rangle \langle v_{01}0| + |v_{01}0\rangle \langle v_{11}1| \\ \rho_{0,1} &= \sum_{x=(x_1,x_2) \in \{0,1\}^2} |v_x \bar{x}_1\rangle \langle v_x \bar{x}_1| \\ &\quad - |v_{00}1\rangle \langle v_{10}0| - |v_{10}0\rangle \langle v_{00}1| - |v_{11}0\rangle \langle v_{01}1| - |v_{01}1\rangle \langle v_{11}0| \\ \rho_{1,0} &= \sum_{x=(x_1,x_2) \in \{0,1\}^2} |v_x x_2\rangle \langle v_x x_2| \\ &\quad + |v_{00}0\rangle \langle v_{01}1| + |v_{01}1\rangle \langle v_{00}0| + |v_{11}1\rangle \langle v_{10}0| + |v_{10}0\rangle \langle v_{11}1| \\ \rho_{1,1} &= \sum_{x=(x_1,x_2) \in \{0,1\}^2} |v_x \bar{x}_2\rangle \langle v_x \bar{x}_2| \\ &\quad - |v_{00}1\rangle \langle v_{01}0| - |v_{01}0\rangle \langle v_{00}1| - |v_{11}0\rangle \langle v_{10}1| - |v_{10}1\rangle \langle v_{11}0|. \end{aligned}$$

where  $\bar{x}_t$  means flipping bit  $x_t$ , i.e.  $\bar{x}_t = 1 - x_t$ .

We look first onto possibilities of Alice's dishonest behaviour. In order to cheat, Alice has to distinguish between density matrices  $\gamma_l = \frac{1}{2}\rho_{l,0} + \frac{1}{2}\rho_{l,1}$ , where  $\gamma_l$  corresponds to  $i = l$ . By examination of the difference of those matrices we get after some calculations that:

$$\gamma_0 - \gamma_1 = \frac{1}{2}|V_S 0\rangle \langle V_A 1| + \frac{1}{2}|V_A 1\rangle \langle V_S 0| - \frac{1}{2}|V_S 1\rangle \langle V_A 0| - \frac{1}{2}|V_A 0\rangle \langle V_S 1|$$

where  $|V_S\rangle = |v_{00}\rangle + |v_{11}\rangle$  and  $|V_A\rangle = |v_{10}\rangle - |v_{01}\rangle$ . We can easily adapt Lemma 1 to show that the advantage  $\delta$  of Alice is at most  $\sum_{l=0}^3 \sigma_l$  where

$$\begin{aligned} \sigma_l &= |tr(H_l(\gamma_0 - \gamma_1)H_l^\dagger)| \leq \sum_{j \in \{0,1\}} \frac{1}{2} |tr(H_l(|V_S(j-1)\rangle\langle V_Aj| + |V_Aj\rangle\langle V_S(j-1)|)H_l^\dagger)| \\ &\leq \sum_{j \in \{0,1\}} (|\langle O_j^l | V_Aj \rangle| \cdot |\langle V_S(1-j) | O_j^l \rangle|) \\ &\leq \sum_{j \in \{0,1\}} |\langle O_j^l | V_Aj \rangle| \end{aligned}$$

and  $|O_j^l\rangle$  is an orthogonal, normalized projection of  $|V_Aj\rangle$  onto subspace  $H_l$ . The second inequality is true because we have  $tr(H_l|V_Aj\rangle\langle \psi|H_l^\dagger) = \langle O_j^l | V_Aj \rangle \langle \psi | O_j^l \rangle$  for every state  $|\psi\rangle$ .

Let  $j_l$  be the index for which  $|\langle O_{j_l}^l | V_Aj_l \rangle| \geq |\langle O_{1-j_l}^l | V_A(1-j_l) \rangle|$ . Clearly,  $\sigma_l \leq 2|\langle O_{j_l}^l | V_Aj_l \rangle|$ . Moreover, we assume that  $\sigma_0 + \sigma_1 \geq \sigma_2 + \sigma_3$ . If this is not the case we could satisfy this condition by altering the strategy  $\mathcal{A}$  of Alice (by appropriate rotation of her basis) in such a way that the definitions of  $H_k$  and  $H_{k+2}$  would swap leaving everything else unchanged.

We look now on the probability of obtaining the correct result by Alice. The probability  $p_0$  of Alice getting outcome  $\beta = 0$  in case of  $\beta = 1$  is at least

$$\begin{aligned} p_0 &\geq \frac{1}{2} \langle O_{j_0}^0 | \rho_{0,1} | O_{j_0}^0 \rangle + \frac{1}{2} \langle O_{j_0}^0 | \rho_{1,1} | O_{j_0}^0 \rangle = \\ &\quad \frac{1}{2} |\langle O_{j_0}^0 | v_{00}1 \rangle - \langle O_{j_0}^0 | v_{01}0 \rangle|^2 + \frac{1}{2} |\langle O_{j_0}^0 | v_{00}1 \rangle - \langle O_{j_0}^0 | v_{10}0 \rangle|^2 \\ &\quad + \frac{1}{2} |\langle O_{j_0}^0 | v_{11}0 \rangle - \langle O_{j_0}^0 | v_{01}1 \rangle|^2 + \frac{1}{2} |\langle O_{j_0}^0 | v_{11}0 \rangle - \langle O_{j_0}^0 | v_{10}1 \rangle|^2. \end{aligned}$$

So, by inequality  $|a-b|^2 + |a-c|^2 \geq \frac{1}{2}|b-c|^2$  we get that

$$\begin{aligned} p_0 &\geq \frac{1}{4} |\langle O_{j_0}^0 | v_{01}0 \rangle - \langle O_{j_0}^0 | v_{10}0 \rangle|^2 + \frac{1}{4} |\langle O_{j_0}^0 | v_{01}1 \rangle - \langle O_{j_0}^0 | v_{10}1 \rangle|^2 \\ &= \frac{1}{4} |\langle O_{j_0}^0 | V_A0 \rangle|^2 + \frac{1}{4} |\langle O_{j_0}^0 | V_A1 \rangle|^2 \geq \frac{1}{16} \sigma_0^2. \end{aligned}$$

Similar calculation of the probability  $p_1$  of getting outcome  $\beta = 1$  in case of  $\beta = 0$  yields that the probability of computing wrong result is at least

$$\Pr[\beta' \neq \beta] = \Pr[\beta \oplus m \neq a_i] \geq \frac{1}{16} (\sigma_0^2 + \sigma_1^2) \geq \frac{1}{256} (\sum_{l=0}^3 \sigma_l)^2.$$

Hence, the lemma holds for the case  $a_0 \oplus a_1 = 0$ .

Since in case of  $a_0 \oplus a_1 = 1$  the reasoning is completely analogous - we exchange only the roles of  $|V_S\rangle$  and  $|V_A\rangle$  and Alice has to know the value of  $\beta \oplus i$  in order to give the correct answer to Bob, the proof is concluded.  $\blacksquare$

To see that quadratical bound imposed by the above lemma can be met, consider  $|\Phi\rangle = \sqrt{1-\varepsilon}|000\rangle + \sqrt{\varepsilon}|110\rangle$ . Intuitively, we label the symmetric and anti-symmetric part of  $|\Phi\rangle$  with 0 and 1. Let  $H_2 = |01\rangle\langle 01|$ ,  $H_3 = 0$ . One can easily calculate that

$$\rho_{0,0} = (1-\varepsilon)|00\rangle\langle 00| + \sqrt{\varepsilon(1-\varepsilon)}(|00\rangle\langle 11| + |11\rangle\langle 00|) + \varepsilon|11\rangle\langle 11|$$

$$\rho_{1,0} = (1-\varepsilon)|00\rangle\langle 00| + \varepsilon|10\rangle\langle 10|$$

and therefore  $\|\rho_{0,0} - \rho_{1,0}\|_t \geq \sqrt{\varepsilon(1-\varepsilon)} - 2\varepsilon$ . So, by Theorem 1 there exists a measurement  $\{H_0, H_1\}$  allowing us to distinguish between those two density matrices with  $\sqrt{\varepsilon(1-\varepsilon)} - 2\varepsilon$  accuracy and moreover  $H_2, H_3 \perp H_0, H_1$  since  $tr(H_2\rho_{0,0}H_2^\dagger) = tr(H_2\rho_{1,0}H_2^\dagger) = 0$ . Now, let  $M = \{H_0, H_1, H_2, H_3\}$  be Alice's measurement. To cheat, we use the following strategy  $\mathcal{A}$  corresponding to her input  $a_0 = a_1 = 0$ . Alice sends  $|\Phi\rangle$  to Bob, after receiving the qubit back she applies the measurement  $M$ . If the outcome is  $H_2$  then she answers  $a_0 \oplus \beta = 1$  to

Bob and sets  $i' = 0$  with probability  $\frac{1}{2}$ , in the other case she sends  $a_0 \oplus \beta = 0$  to Bob and according to the outcome being 0 or 1 she sets  $i' = 0$  ( $i' = 1$ ).

To see that this strategy gives correct result with probability greater than  $1 - \varepsilon$  we should note that probability of outcome  $H_2$  in case of  $\beta = 0$  is 0 and in case of  $\beta = 1$  is  $1 - \varepsilon$ . Therefore, since  $\beta = 0$  with probability  $\frac{1}{2}$ , our advantage in determining the input of Bob is greater than  $\frac{1}{2}\sqrt{\varepsilon} - \frac{3}{2}\varepsilon$ .

### 3.2 Malicious Bob

Now, we analyze Bob's possibility of cheating.

**Lemma 3** *Let Alice and Bob perform Protocol 2. Assume Alice is honest and deposits bits  $a_0, a_1$ , with  $\Pr[a_i = 0] = 1/2$ . Then for every strategy used by Bob and all values  $a'_0, a'_1$  which Bob learns about  $a_0, a_1$ , it holds that: for all  $i \in \{0, 1\}$*

$$\text{if } \Pr_{a_0, a_1 \in R\{0,1\}}[a'_i = a_i] \geq 1 - \varepsilon^2 \text{ then } \Pr_{a_0, a_1 \in R\{0,1\}}[a'_{1-i} = a_{1-i}] \leq 1/2 + 16\sqrt{2}\varepsilon.$$

*Proof:* Consider some malicious strategy  $\mathcal{B}$  of Bob. Wlog we may assume that the probability of  $a'_0 = a_0$  is greater than the probability of  $a'_1 = a_1$ . Our aim is to show that

$$\text{if } \Pr_{a_0, a_1 \in R\{0,1\}}[a'_0 \neq a_0] \leq \varepsilon^2 \text{ then } \Pr_{a_0, a_1 \in R\{0,1\}}[a'_1 = a_1] \leq 1/2 + 16\sqrt{2}\varepsilon.$$

Strategy  $\mathcal{B}$  can be think of as a two step process. First a unitary transformation  $U$  is acting on  $|\Phi_{a_0, a_1, h}\rangle = |v\rangle \otimes R_\alpha|a_1 \oplus h\rangle \otimes R_\alpha|a_0 \oplus h\rangle$ , where  $v$  is an ancillary state<sup>2</sup>. Next the last qubit of  $U(|\Phi_{a_0, a_1, h}\rangle)$  is sent to Alice<sup>3</sup>, she performs step 3 on these qubit and sends the classical bit  $m$  back to Bob. Upon receiving  $m$ , Bob executes the second part of his attack: he performs some arbitrary measurement  $\{H_0, H_1, H_2, H_3\}$ , where  $H_0$  ( $H_1$ ) corresponds to Bob's belief that  $a_0 = 0, a_1 = 0$  (resp.  $a_0 = 0, a_1 = 1$ ) and  $H_2$  ( $H_3$ ) corresponds to  $a_0 = 1$  and  $a_1 = 0$  (resp.  $a_0 = 1$  and  $a_1 = 1$ ). In other words, outcome corresponding to  $H_{2l+k}$  implies  $a'_0 = l$  and  $a'_1 = k$ .

The unitary transformation  $U$  can be described by a set of vectors  $\{V_k^{l,j}\}$  such that  $U(|v\rangle \otimes |l, j\rangle) = |V_0^{l,j}\rangle \otimes |0\rangle + |V_1^{l,j}\rangle \otimes |1\rangle$ . Or alternatively in diagonal basis, by a set of vectors  $\{W_k^{l,j}\}$  such that  $U(|v\rangle \otimes |l_x, j_x\rangle) = |W_0^{l,j}\rangle \otimes |0_x\rangle + |W_1^{l,j}\rangle \otimes |1_x\rangle$ .

We present now, an intuitive, brief summary of the proof. Informally, we can think of  $U$  as about some kind of disturbance of the qubit  $R_\alpha|a_0 \oplus h\rangle$  being sent back to Alice. First, we will show that in order to cheat Bob's  $U$  has to accumulate after Step 2, till the end of the protocol, some information about the value of  $a_0 \oplus h$  hidden in this qubit. On the other hand, to get the proper result i.e. the value of  $a_0$ , this qubit's actual information about encoded value has to be disturbed at the smallest possible degree. That implies for Bob a necessity of some sort of cloning that qubit, which turns out to impose the desired bounds on possible cheating. We show this by first reducing the task of cloning to one where no additional hint in the form of  $R_\alpha|a_1 \oplus h\rangle$  is provided and then an analysis of this simplified process. Therefore, the proof indicates that the hardness of cheating the protocol is contained in the necessity of cloning, which gives us a sort of quantitative non-cloning theorem. Although, it seems to concern only our particular implementation of the protocol, we believe that this scenario is useful enough to be of independent interests.

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<sup>2</sup>Note that this does not restrict Bob's power. Particularly, when Bob tries to make a measurement in the first step then using a standard technique we can move this measurement to the second step.

<sup>3</sup>We can assume wlog that the last qubit is sent since  $U$  is arbitrary

We analyze first Bob's information gain about  $a_1$ . Wlog we may assume that Bob can distinguish better between two values of  $a_1$  if  $a_0 = 0$ . That is

$$\Pr_{a_1 \in R\{0,1\}}[a'_1 = a_1 | a_0 = 0] \geq \Pr_{a_1 \in R\{0,1\}}[a'_1 = a_1 | a_0 = 1].$$

Let now  $\rho_{j,k,l}$  be a density matrix of the system before Bob's final measurement, corresponding to  $\alpha = j \cdot \frac{1}{2}$ ,  $h = k$ ,  $a_1 = l$  and  $a_0 = 0$ . The advantage  $\delta$  of Bob in this case (i.e.  $\delta$  such that  $\Pr[a'_1 = a_1 | a_0 = 0] = 1/2 + \delta$ ) can be estimated by Lemma 1 by Bob's ability to distinguish between the following density matrices:

$$\begin{aligned} \frac{1}{4}(\rho_{0,0,0} + \rho_{1,0,0} + \rho_{0,1,0} + \rho_{1,1,0}) & \quad (\text{case } a_1 = 0), \text{ and} \\ \frac{1}{4}(\rho_{0,0,1} + \rho_{1,0,1} + \rho_{0,1,1} + \rho_{1,1,1}) & \quad (\text{case } a_1 = 1). \end{aligned}$$

Using the triangle inequality we get that for the measurement  $\mathcal{O}$  performed by Bob

$$\delta \leq \frac{1}{8}(|\rho_{0,0,0}^{\mathcal{O}} - \rho_{0,1,1}^{\mathcal{O}}|_1 + |\rho_{1,0,0}^{\mathcal{O}} - \rho_{1,0,1}^{\mathcal{O}}|_1 + |\rho_{0,1,0}^{\mathcal{O}} - \rho_{0,0,1}^{\mathcal{O}}|_1 + |\rho_{1,1,0}^{\mathcal{O}} - \rho_{1,1,1}^{\mathcal{O}}|_1). \quad (1)$$

Each component corresponds to different values of  $\alpha$  and  $h \oplus a_1$ . And each component is symmetric to the other in such a way that there exists a straight-forward local transformation for Bob (i.e. appropriate rotation of the computational basis on one or both qubits) which transform any of above components onto another. So, we can assume wlog that the advantage in distinguishing between  $\rho_{0,0,0}$  and  $\rho_{0,1,1}$   $\delta_0 = |\rho_{0,0,0}^{\mathcal{O}} - \rho_{0,1,1}^{\mathcal{O}}|_1$  is the maximum component in the right-hand side of the inequality (1) and therefore we have  $\delta \leq \frac{1}{2}\delta_0$ . Let, for short,  $\gamma_0 = \rho_{0,0,0}$  and  $\gamma_1 = \rho_{0,1,1}$ . One can easily calculate that

$$\gamma_0 = |0\rangle\langle 0| \otimes |V_0^{00}\rangle\langle V_0^{00}| + |1\rangle\langle 1| \otimes |V_1^{00}\rangle\langle V_1^{00}| \quad (2)$$

$$\gamma_1 = |0\rangle\langle 0| \otimes |V_1^{01}\rangle\langle V_1^{01}| + |1\rangle\langle 1| \otimes |V_0^{01}\rangle\langle V_0^{01}|. \quad (3)$$

As we can see to each value of  $m$  in above density matrices corresponds a pair of vectors which are critical for Bob's cheating. I.e. the better they can be distinguishable by his measurement the greater is his advantage. But, as we will see later, this fact introduces perturbation of the indication of the value of  $a_0$ .

First, we take a look on the measurements  $H_0, H_1$  performed by Bob. Let us define  $\sigma_{2m+p}$  for  $p, m \in \{0, 1\}$  as follows

$$\sigma_{2m+p} = \begin{cases} |tr(H_p|0V_p^{0p}\rangle\langle 0V_p^{0p}|H_p^\dagger) - tr(H_p|0V_{1-p}^{0(1-p)}\rangle\langle 0V_{1-p}^{0(1-p)}|H_p^\dagger)| & \text{if } m = 0, \\ |tr(H_p|1V_{1-p}^{0p}\rangle\langle 1V_{1-p}^{0p}|H_p^\dagger) - tr(H_p|1V_p^{0(1-p)}\rangle\langle 1V_p^{0(1-p)}|H_p^\dagger)| & \text{if } m = 1. \end{cases}$$

Let for  $m = 0$ ,  $p_0 \in \{0, 1\}$  be such that  $\sigma_{p_0} \geq \sigma_{1-p_0}$  and similarly, for  $m = 1$  let  $p_1 \in \{0, 1\}$  be such that  $\sigma_{2+p_1} \geq \sigma_{2+(1-p_1)}$ . Then we get

$$\begin{aligned} |\gamma_0^{\mathcal{O}} - \gamma_1^{\mathcal{O}}|_1 &= \sum_{t=0}^3 |tr(H_t\gamma_0 H_t^\dagger) - tr(H_t\gamma_1 H_t^\dagger)| \\ &\leq 2(\sigma_{p_0} + \sigma_{2+p_1}) + \sum_{t=2}^3 |tr(H_t\gamma_0 H_t^\dagger) - tr(H_t\gamma_1 H_t^\dagger)|. \end{aligned}$$

We should see first that the second term in the above sum corresponds to advantage in distinguishing between two values of  $a_1$  by measurement  $H_2, H_3$  in case of  $a_0 = 0$ . But those subspaces reflect Bob's belief that  $a_0 = 1$ . Therefore, we have that

$$\sum_{t=2}^3 |tr(H_t\gamma_0 H_t^\dagger) - tr(H_t\gamma_1 H_t^\dagger)| \leq \Pr_{a_0, a_1 \in R\{0,1\}}[a'_0 \neq a_0 | a_0 = 0].$$

So, we can neglect this term because it is of the order of the square of the advantage (if not then our lemma would be proved). Hence we get:  $\frac{\delta_0}{2} \leq \sigma_{p_0} + \sigma_{2+p_1}$ .

Now, we define projection  $O_m$  as follows. For  $m = 0$  let  $O_0$  be the normalized orthogonal projection of  $|0V_{p_0}^{0p_0}\rangle$  onto the subspace  $H_{p_0}$  if

$$\text{tr}(H_{p_0}|0V_{p_0}^{0p_0}\rangle\langle 0V_{p_0}^{0p_0}|H_{p_0}^\dagger) \geq \text{tr}(H_{p_0}|0V_{1-p_0}^{0(1-p_0)}\rangle\langle 0V_{1-p_0}^{0(1-p_0)}|H_{p_0}^\dagger).$$

Otherwise, let  $O_0$  be the normalized orthogonal projection of  $|0V_{1-p_0}^{0(1-p_0)}\rangle$  onto  $H_{p_0}$ . Analogously, we define  $O_1$  as a normalized orthogonal projection of  $|1V_{1-p_1}^{0p_1}\rangle$  onto the subspace  $H_{p_1}$  if

$$\text{tr}(H_{p_1}|1V_{1-p_1}^{0p_1}\rangle\langle 1V_{1-p_1}^{0p_1}|H_{p_1}^\dagger) \geq \text{tr}(H_{p_1}|1V_{p_1}^{0(1-p_1)}\rangle\langle 1V_{p_1}^{0(1-p_1)}|H_{p_1}^\dagger)$$

else  $O_1$  is a normalized orthogonal projection of  $|1V_{p_1}^{0(1-p_1)}\rangle$  onto  $H_{p_1}$ . Hence we get

$$\sigma_{p_0} \leq |\langle 0V_{p_0}^{0p_0}|O_0\rangle|^2 - |\langle 0V_{1-p_0}^{0(1-p_0)}|O_0\rangle|^2, \quad \sigma_{2+p_1} \leq |\langle 1V_{1-p_1}^{0p_1}|O_1\rangle|^2 - |\langle 1V_{p_1}^{0(1-p_1)}|O_1\rangle|^2.$$

We would like now to investigate the probability of obtaining the correct result. Recall that  $\Pr[a_1 = 0] = \frac{1}{2}$ . We should first note that the density matrices corresponding to initial configuration of the second qubit  $R_\alpha|a_1 \oplus h\rangle$  is now exactly  $\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$  even if we know  $h$  and  $\alpha$ . So, from the point of view of the protocol those two configurations are indistinguishable. Therefore, we can substitute the second qubit from the initial configuration with a random bit  $r$  encoded in perpendicular basis and the probability of obtaining proper result is unchanged. We analyze the probability of computing the correct result in case of  $r = 0$ . Note, that the vectors  $\{V_k^{0,j}\}_{k,j}$  still describe  $U$ , but vectors  $\{W_k^{0j}\}_{k,j}$  are different, defined by  $U$  acting now on initial configuration  $|v\rangle \otimes |0\rangle \otimes R_\alpha|j\rangle$ , with  $\alpha = \frac{1}{2}$ . We investigate the correspondence between  $\{V_k^{0,j}\}_{k,j}$  and the new vectors. For  $j = 0$  we have:

$$\begin{aligned} U(|v00\rangle\langle v00|) &= \frac{1}{\sqrt{2}}U(|v00\rangle - |v01\rangle) = \frac{1}{\sqrt{2}}(V_0^{00}|0\rangle + V_1^{00}|1\rangle - V_0^{01}|0\rangle - V_1^{01}|1\rangle) \\ &= \frac{1}{2}((V_0^{00} - V_1^{00} - V_0^{01} + V_1^{01})|0\rangle\langle 0| + (V_0^{00} + V_1^{00} - V_0^{01} - V_1^{01})|1\rangle\langle 1|). \end{aligned}$$

Similarly, for  $j = 1$  we have:

$$\begin{aligned} U(|v01\rangle\langle v01|) &= \frac{1}{\sqrt{2}}U(|v00\rangle + |v01\rangle) = \frac{1}{\sqrt{2}}(V_0^{00}|0\rangle + V_1^{00}|1\rangle + V_0^{01}|0\rangle + V_1^{01}|1\rangle) \\ &= \frac{1}{2}((V_0^{00} - V_1^{00} + V_0^{01} - V_1^{01})|0\rangle\langle 0| + (V_0^{00} + V_1^{00} + V_0^{01} + V_1^{01})|1\rangle\langle 1|). \end{aligned}$$

Thus, let us denote these vectors by

$$\begin{aligned} \widetilde{W}_0^{00} &= \frac{1}{2}((V_0^{00} + V_1^{00}) - (V_0^{01} + V_1^{01})), & \widetilde{W}_1^{00} &= \frac{1}{2}((V_0^{00} - V_1^{00}) - (V_0^{01} - V_1^{01})), \\ \widetilde{W}_0^{01} &= \frac{1}{2}((V_0^{00} - V_1^{01}) + (V_0^{01} - V_1^{00})), & \widetilde{W}_1^{01} &= \frac{1}{2}((V_0^{00} + V_1^{01}) + (V_0^{01} + V_1^{00})). \end{aligned}$$

In order to obtain the correct result Bob has to distinguish between the density matrices corresponding to two values of  $a_0$ . In particular, he has to distinguish between density matrices  $\gamma'_0, \gamma'_1$  corresponding to two possible values of  $a_0$  knowing that  $m = 0$ . These density matrices are:

$$\gamma'_0 = \frac{1}{4}|0\rangle\langle 0| \otimes (|V_0^{00}\rangle\langle V_0^{00}| + |V_1^{01}\rangle\langle V_1^{01}| + |\widetilde{W}_0^{00}\rangle\langle \widetilde{W}_0^{00}| + |\widetilde{W}_1^{01}\rangle\langle \widetilde{W}_1^{01}|), \quad (4)$$

$$\gamma'_1 = \frac{1}{4}|0\rangle\langle 0| \otimes (|V_0^{01}\rangle\langle V_0^{01}| + |V_1^{00}\rangle\langle V_1^{00}| + |\widetilde{W}_0^{01}\rangle\langle \widetilde{W}_0^{01}| + |\widetilde{W}_1^{00}\rangle\langle \widetilde{W}_1^{00}|). \quad (5)$$

Now, the probability of failure i.e. the probability that in case of  $m = 0$  Bob's measurement indicates that  $a_0 = 0$  if in fact it is  $a_0 = 1$ , is at least

$$\text{tr}(H_{p_0}\gamma'_1 H_{p_0}^\dagger) \geq \text{tr}(|O_0\rangle\langle O_0|\gamma'_1) = \frac{1}{4}(|\langle 0V_0^{01}|O_0\rangle|^2 + |\langle 0V_1^{00}|O_0\rangle|^2 + |\langle 0\widetilde{W}_0^{01}|O_0\rangle|^2 + |\langle 0\widetilde{W}_1^{00}|O_0\rangle|^2).$$

But since the fact that

$$\widetilde{W}_0^{01} = \frac{1}{2}((V_0^{00} - V_1^{01}) + (V_0^{01} - V_1^{00})), \quad \widetilde{W}_1^{00} = \frac{1}{2}((V_0^{00} - V_1^{01}) - (V_0^{01} - V_1^{00})),$$

and the parallelogram law ( $|a+b|^2 + |a-b|^2 = 2|a|^2 + 2|b|^2$ ), we have that this probability is at least

$$\begin{aligned} \frac{1}{4}(|\langle 0\widetilde{W}_0^{01}|O_0\rangle|^2 + |\langle 0\widetilde{W}_1^{00}|O_0\rangle|^2) &\geq \frac{1}{8}|\langle 0V_0^{00}|O_0\rangle - \langle 0V_1^{01}|O_0\rangle|^2 \\ &\geq \frac{1}{32}(|\langle 0V_0^{00}|O_0\rangle| - |\langle 0V_1^{01}|O_0\rangle|)^2(|\langle 0V_0^{00}|O_0\rangle| + |\langle 0V_1^{01}|O_0\rangle|)^2 \\ &\geq \frac{1}{32}(|\langle 0V_0^{00}|O_0\rangle|^2 - |\langle 0V_1^{01}|O_0\rangle|^2)^2 \geq \frac{\sigma_{p_0}^2}{32}. \end{aligned}$$

Similarly we analyze density matrices  $\gamma''_0$ ,  $\gamma''_1$  corresponding to two possible values of  $a_0$  knowing that  $m = 1$ . These density matrices are equal to resp.  $\gamma'_1$  and  $\gamma'_0$  after changing  $|0\rangle\langle 0|$  to  $|1\rangle\langle 1|$ . Now, by repeating completely analogous estimation of failure's probability with usage of vectors  $|V_0^{01}\rangle$ ,  $|V_1^{00}\rangle$ ,  $|\widetilde{W}_0^{00}\rangle$  and  $|\widetilde{W}_1^{01}\rangle$ , we get that this probability is at least  $\frac{\sigma_{p_1}^2}{32}$ . Therefore, since the vectors involved in imposing failure in both cases are distinct, we conclude that  $\Pr_{a_1 \in R\{0,1\}}[a'_0 \neq a_0 | r = 0] \geq \frac{\sigma_{p_0}^2 + \sigma_{p_1}^2}{32}$ . Hence we have

$$\begin{aligned} \Pr_{a_1 \in R\{0,1\}}[a'_0 \neq a_0] &= \frac{1}{2}\Pr_{a_1 \in R\{0,1\}}[a'_0 \neq a_0 | r = 0] + \frac{1}{2}\Pr_{a_1 \in R\{0,1\}}[a'_0 \neq a_0 | r = 1] \\ &\geq \frac{\sigma_{p_0}^2 + \sigma_{p_1}^2}{64} \geq \frac{\delta^2}{128} \end{aligned}$$

and the lemma is proved.

Finally, it is worth mentioning that the value of  $m$  doesn't need to be correlated in any way with value of  $a_i$ . That is, Bob by using entanglement (for instance, straightforward use of Bell states) can make the value of  $m$  independent of  $a_i$  and still acquire perfect knowledge about  $a_i$ . He uses simple error-correction to know whether  $m = a_i$  or  $m = 1 - a_i$ . His problems with determining whether flip has occurred, start only when he wants additionally to accumulate some information about the value of  $a_i \oplus h$ .  $\blacksquare$

To see that this quadratical bound can be achieved consider the following cheating strategy. Let  $U^*$  be such that  $U^*(|v\rangle \otimes |l, j\rangle) = |v_j\rangle \otimes |l, j\rangle$ . So,  $|V_j^{l,j}\rangle = |v_j\rangle \otimes |l\rangle$  and  $|V_{1-j}^{l,j}\rangle = 0$ . Moreover, let  $\langle v_0|v_1\rangle = \sqrt{1-\varepsilon}$ . As we can see, usage of  $U^*$  accumulates some information about value of  $j = a_0 \oplus h$  by marking it with two non-parallel (therefore possible to distinguish) vectors in Bob's system. We do now the following. We use  $U^*$  on  $|v\rangle \otimes R_\alpha|a_1 \oplus h\rangle \oplus R_\alpha|a_0 \oplus h\rangle$  and send the last qubit to Alice. When we get the message  $m$  which is exactly  $a_0$  with probability<sup>4</sup> of order  $1 - \varepsilon$ , we make an optimal measurement to distinguish between  $v_0$  and  $v_1$ . By Theorem 1 this optimal measurement has advantage of order  $\sqrt{\varepsilon}$ . So, after getting the outcome  $j'$ , we know that  $\Pr[j' = a_0 \oplus h] \geq \frac{1}{2} + \Omega(\sqrt{\varepsilon})$  and we can simply compute the value of  $h' = m \oplus j'$ . Having such knowledge about the value of  $h'$  we can distinguish between values of  $a_1$  encoded in the second qubit  $R_\alpha|a_1 \oplus h\rangle$  with the advantage proportional to  $\Omega(\sqrt{\varepsilon})$ .

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<sup>4</sup>This can be easily computed - the perturbation arises when  $\alpha = \frac{1}{2}$ .

## 4 Cheat Sensitive Quantum Bit Commitment

We recall first a formal definition of the binding and sealing property of a quantum bit commitment. We follow here the definition by Aharonov et al. [2]. Let us start with the binding property. Assume Alice follows the bit commitment protocol and Bob is arbitrarily. During the depositing phase Bob and Alice compute in some rounds a super-position  $|\psi_{AB}\rangle$  with two quantum registers: one keeping by Bob and one by Alice. After a communication phase Bob either uses a strategy trying to convince Alice to 0 or a strategy to convince Alice to 1. Depending on the results of the computations Alice decides to one the values  $v_B \in \{0, 1, \text{err}\}$ ; In case  $v_B = \text{err}$  he rejects the protocol. Let  $p_i$  be the probability that Alice decides  $v_B = i$ , and  $p_{\text{err}}$  be the probability that Alice decides  $v_B = \text{err}$ , when Bob uses strategy 0. Analogously, denote the probabilities  $q_0, q_1, q_{\text{err}}$  for Bob's strategy 1. A protocol is  $(\delta, \varepsilon)$ -binding if whenever Alice is honest, for any Bob's strategy it is true: if  $p_{\text{err}}, q_{\text{err}} \leq \varepsilon$  then  $|p_0 - q_0|, |p_1 - q_1| \leq \delta$ . A bit commitment protocol is  $(\delta, \varepsilon)$ -sealing, if whenever Bob is honest and deposits a bit  $b$  s.t.  $\Pr[b = 0] = 1/2$ , for any Alice's strategy and a value  $c$  Alice learns, it holds that: if  $\Pr_{b \in R^{\{0,1\}}}[\text{Bob detects error}] \leq \varepsilon$  then  $\Pr_{b \in R^{\{0,1\}}}[c = b] \leq 1/2 + \delta$ . The probability is taken over  $b$  taken uniformly from  $\{0, 1\}$  and the protocol.

**Theorem 3** *Using Protocol 2 as a black-box for computing OT, Protocol 1 is an  $(4\sqrt{\varepsilon}, \varepsilon)$ -sealing. Moreover, there exists a constant  $\lambda > 0$  such that for all strategies Bob uses it holds  $\max\{p_{\text{err}}, q_{\text{err}}\} > \lambda$ , where  $p_{\text{err}}$  ( $q_{\text{err}}$ ) denotes the probability that Alice decides error when Bob uses strategy for 0 (1 resp.).*

*Sketch of the proof:* First, we note that in both calls to the OT function the inputs that come into play in this executions are completely uncorrelated from the point of view of both Alice and Bob. So, we can analyze them distinctly.

To see that this protocol is sealing we note that Alice in each call to OT function has to take into account that with probability  $\frac{1}{2}$  Bob will check whether she knows what actually he has received during execution of this protocol. Moreover her cheating is effective only if it is not checked, so only with probability of  $\frac{1}{2}$ . By Lemma 2, if a strategy allows her to distinguish between possible values of  $b'$  with advantage greater than  $4\sqrt[3]{2\varepsilon}$  then  $\Pr_{b' \in R^{\{0,1\}}}[v_c \neq \text{OT}((a_{2c}, a_{2c+1}), b')] \geq \varepsilon$ .

In case of binding, we first notice that it is only useful for Bob to cheat in some particular OT execution, chosen previously by Bob, which is used in the revealing phase for the binding test. So wlog assume Bob cheats in the second OT execution and that in the last step of the depositing stage he reveals  $c = 0$ . Let  $a'_3, a'_4$ , resp. denote the predicted values. Using the notation given in the definition of the binding property we get that  $p_{\text{err}} = \Pr[a'_3 \neq a_3]$ ,  $p_0 = \Pr[a'_3 = a_3]$ , and  $p_1 = 0$ . Similarly we have  $q_{\text{err}} = \Pr[a'_4 \neq a_4]$ ,  $q_0 = 0$ , and  $q_1 = \Pr[a'_4 = a_4]$ . Now by Lemma 3 we get that if  $\Pr[a'_i \neq a_i] \leq \varepsilon^2$  then  $\Pr[a'_{1-i} \neq a_{1-i}] \geq 1/2 - 16^2\varepsilon$  and for some constant  $\lambda > 0$  it follows that  $\max\{\Pr[a'_i \neq a_i], \Pr[a'_{1-i} \neq a_{1-i}]\} > \lambda$ . ■

## 5 Concluding Remark

In this paper a weak variant of quantum bit commitment is investigated. We give quantum bit commitment scheme that is simultaneously binding and sealing and we show that if a malicious Alice gains some information about the committed bit  $b$  then Bob detects this with a probability  $\Omega(\varepsilon^2)$ . When Bob cheats then Alice's probability of detecting the cheating is greater than a constant  $\lambda > 0$ . Using our bounds we get that the value is very small and an interesting task would be to improve the constant.

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